Episode 11

Free Vibration of Un-Damped Systems
Part 1

ENGN0040: Dynamics and Vibrations
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Topics for todays class

Free Vibration of Un-damped Systems

1. Examples of vibrations and their consequences
2. Typical vibration response, simple harmonic motion
3. The harmonic oscillator
4. Natural frequencies and vibration (normal) modes
5. Counting degrees of freedom and vibration modes

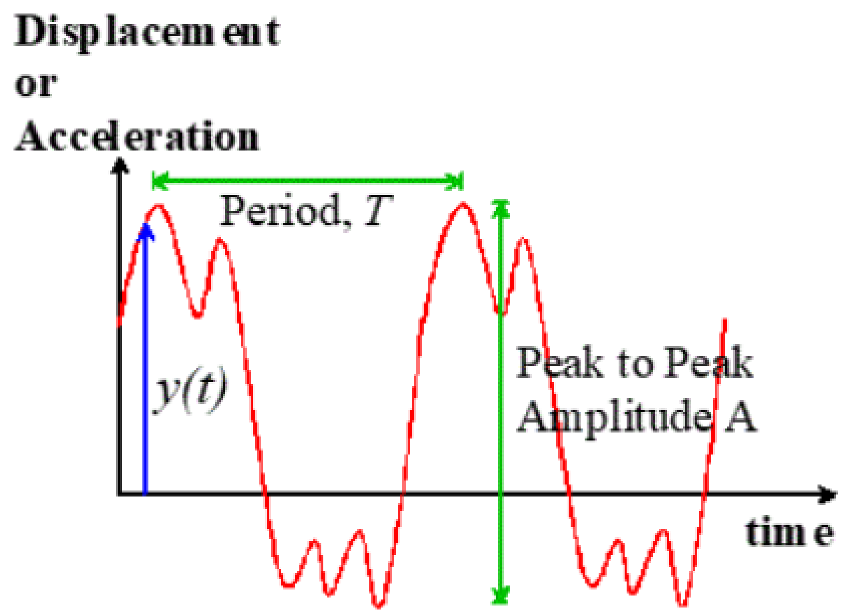
5 VIBRATIONS

5.1 Features of a typical vibration response

Time for 1 cycle (period) T

Frequency $f = 1/T$ (Hertz)

Angular frequency $\omega = 2\pi f$
(rad/sec)



Peak-peak amplitude $y_{max} - y_{min}$

RMS amplitude

$$\langle y \rangle = \left\{ \frac{1}{T} \int_0^T (y - \bar{y})^2 \, dt \right\}^{1/2} \qquad \bar{y} = \frac{1}{T} \int_0^T y \, dt$$

5.2 Simple Harmonic Motion

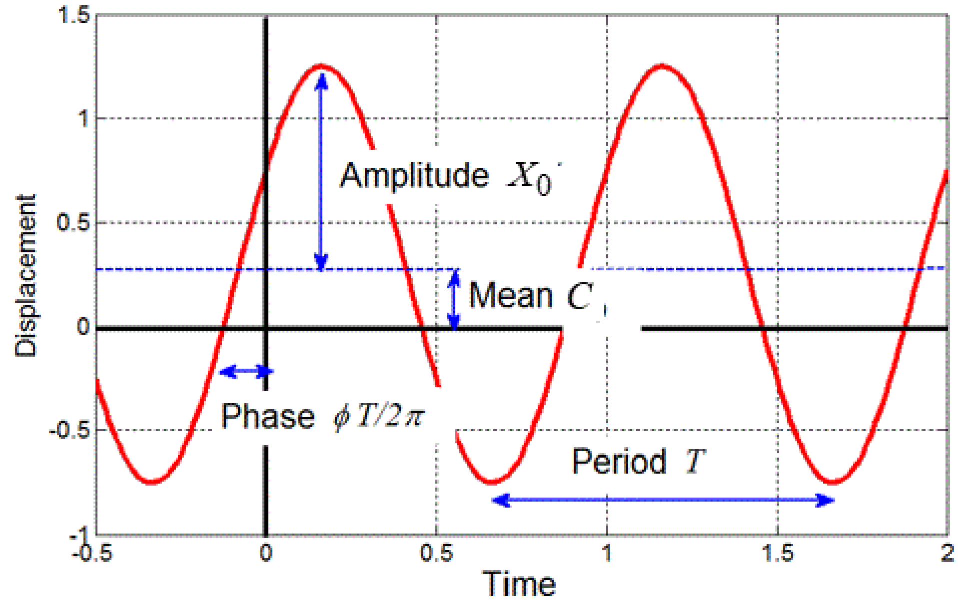
$$x = C + X_0 \sin(\omega t + \phi)$$

ω : angular frequency

$\omega = 2\pi / T$

X_0 : amplitude

ϕ : phase



Velocity

$$v = dx/dt = \omega X_0 \cos(\omega t + \phi)$$
$$= V_0 \cos(\omega t + \phi)$$

Acceleration

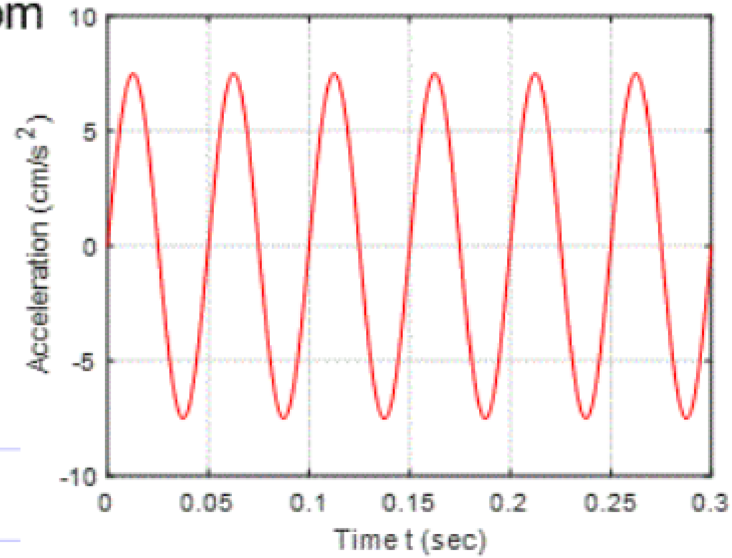
$$a = dv/dt = -\omega V_0 \sin(\omega t + \phi) = -\omega^2 X_0 \sin(\omega t + \phi)$$
$$= -A_0 \sin(\omega t + \phi)$$

Amplitude formulas

$$V_0 = \omega X_0 \qquad A_0 = \omega V_0 = \omega^2 X_0$$

5.3: Example: The figure shows a vibration measurement from an accelerometer. Calculate

(a) The amplitude of the acceleration
(b) The period of oscillation
(c) The angular frequency of oscillation (in rad/s)
(d) The amplitude of the velocity
(e) The amplitude of the displacement



From graph: $A_0 = 7.5\ cm/s^2$

Period: 6 cycles in 0.3s $\Rightarrow T = 0.3/6 = 0.05s$

$\omega = 2\pi/T = 40\pi$ rad/s (or $f = 1/T$ cycles/s)

Formula $V_0 = A_0/\omega = 7.5/(40\pi)$ cm/s

$X_0 = V_0/\omega = 7.5/(40\pi)^2$ cm

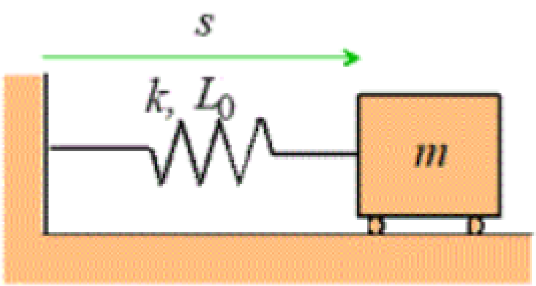
5.4 Free vibration of un-damped systems

"Free" ⇒ no time dependent external forces
"Undamped" ⇒ energy is constant

5.4.1 Vibration of a 1 degree of freedom system

1 DOF ⇒ motion can be described by 1 coord

Canonical vibration problem: The spring-mass system is released with speed v_0 from position s_0 at time $t=0$. Find $s(t)$

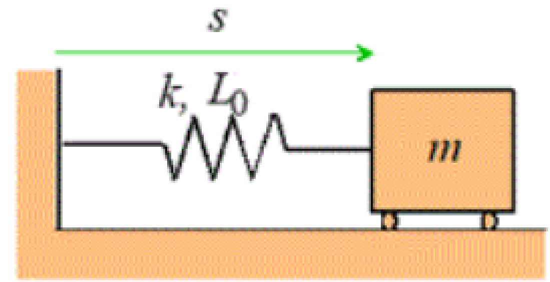


"Harmonic Oscillator"

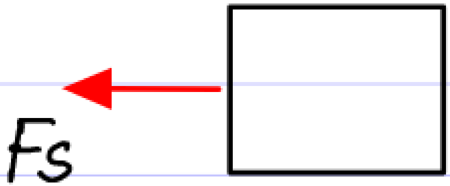
Approach

(1) $F = ma$
(2) Differential eq for s
(3) Look up solution in tables

Canonical vibration problem: The spring-mass system is released with speed v_0 from position s_0 at time $t=0$. Find $s(t)$



FBD



$$F = ma \Rightarrow -F_s = m\frac{d^2s}{dt^2}$$

Spring Fomula $\quad F_s = k(s-L_0)$

$$\Rightarrow -k(s-L_0) = m\frac{d^2s}{dt^2}$$

Rearrange:

Constants on RHS

$$\frac{m}{k}\frac{d^2s}{dt^2} + s = L_0$$

No coefficient in front of s

Look up solution in tables

List of standard ODEs for vibration problems

Case I $$\boxed{\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+x=C}$$

Our eq: $$\frac{m}{k}\frac{d^2s}{dt^2}+s=L_0$$

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+x=C$$

Case II $$\frac{1}{\alpha^2}\frac{d^2x}{dt^2}-x=-C$$

Case III $$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C$$

Case IV $$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C+KF(t) \text{ with } F(t)=F_0\sin\omega t$$

Case V $$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C+K\left(y+\frac{2\varsigma}{\omega_n}\frac{dy}{dt}\right) \text{ with } y(t)=Y_0\sin\omega t$$

Case VI $$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C-\frac{K}{\omega_n^2}\frac{d^2y}{dt^2} \text{ with } y(t)=Y_0\sin\omega t$$

Case VII $$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx}{dt}+x=K\left(\frac{\lambda^2}{\omega_n^2}\frac{d^2y}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dy}{dt}+y\right) \text{ with } y(t)=Y_0\sin\omega t$$

Hence $s \equiv x$ $\quad L_0 \equiv C$ $\quad \frac{1}{\omega_n^2}=\frac{m}{k} \Rightarrow \boxed{\omega_n=\sqrt{\frac{k}{m}}}$

Solution to Case I (From handout)

Equation $\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+x=C$ Initial Conditions $x=x_0$ $\frac{dx}{dt}=v_0$ $t=0$

Solution $x=C+X_0\sin(\omega_n t+\phi)$ $X_0=\sqrt{(x_0-C)^2+v_0^2/\omega_n^2}$ $\phi=\cos^{-1}\left(\frac{v_0/\omega_n}{\sqrt{(x_0-C)^2+v_0^2/\omega_n^2}}\right)$

Or $x(t)=C+(x_0-C)\cos\omega_n t+\frac{v_0}{\omega_n}\sin\omega_n t$

Hence $\boxed{s = L_0 + \bar{X}_0 \sin(\omega_n t + \phi)}$

$$\bar{X}_0 = \sqrt{(s_0 - L_0)^2 + V_0^2/\omega_n^2} \qquad \phi = \cos^{-1}\left(\frac{V_0}{\omega_n \bar{X}_0}\right)$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

Simple Hamonic Motion @ frequency ω_n

How to solve the case I vibration equation

Solve: $\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+x=C \qquad x=x_0 \quad \frac{dx}{dt}=v_0 \quad t=0$

General approaches: (1) Guess solution, substitute into ODE; or

(2) Rearrange into form with known solution (eg transforms)

Use (1) here – vibration problem so $\boxed{\text{guess} \quad x=A\sin\lambda t+B\cos\lambda t+C}$

Substitute into ODE: $\frac{1}{\omega_n^2}\left(-\lambda^2 A\sin\lambda t-\lambda^2 B\cos\lambda t\right)+A\sin\lambda t+B\cos\lambda t+C=C$

$$\Rightarrow\left(-\frac{\lambda^2}{\omega_n^2}+1\right)(A\sin\lambda t+B\cos\lambda t)=0$$

$$-\frac{\lambda^2}{\omega_n^2}+1=0 \longrightarrow \boxed{\lambda=\omega_n}$$

Find A,B from initial conditions:

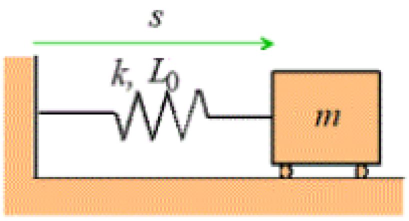
$$x=A\sin(0)+B\cos(0)+C=x_0 \qquad (t=0) \qquad \frac{dx}{dt}=A\omega_n\cos(0)-B\omega_n\sin(0)=v_0$$

$$\longrightarrow \quad B+C=x_0, \quad A\omega_n=v_0 \quad \longrightarrow \quad B=x_0-C, \quad A=v_0/\omega_n$$

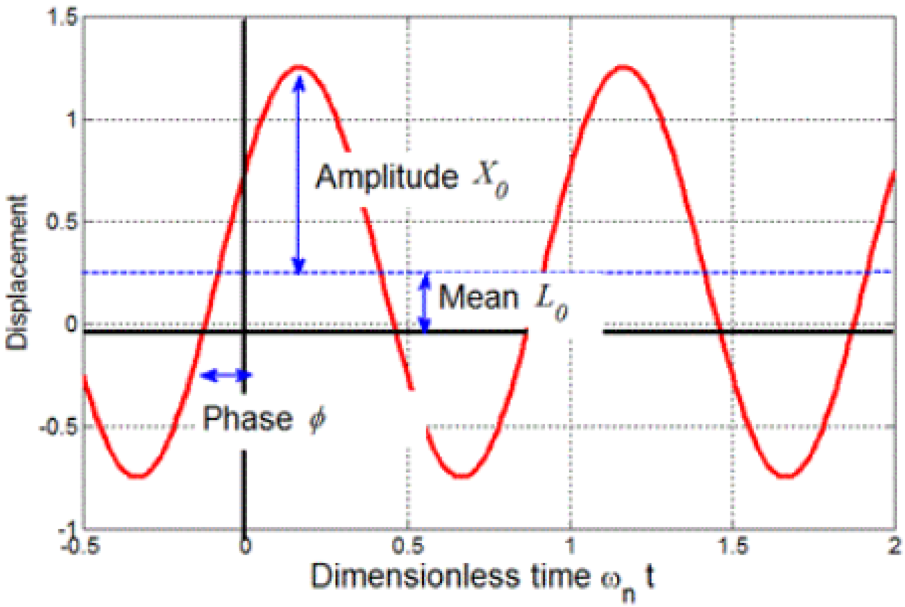
$$\boxed{x=\frac{v_0}{\omega_n}\sin\omega_n t+(x_0-C)\cos\omega_n t+C}$$

Summary

Canonical vibration problem: The spring-mass system is released with speed v_0 from position s_0 at time $t=0$. Find $s(t)$



$$s = L_0 + X_0 \sin(\omega_n t + \phi)$$



Notes:

(1) Vibration frequency
$\omega_n = \sqrt{k/m}$ is
a property of system

(2) Amplitude depends on initial conditions

(3) Phase depends on initial conditions

These apply to all undamped 1 DOF stable systems

5.4.2 Vibration of systems with many DOF

(1) In general motion will not be harmonic
(2) There are special initial conditions that cause harmonic vibrations
(3) Usually each special initial displacement causes vibration at a different frequency
(4) Special frequencies are called "natural frequencies"
(5) Special initial displacements are called "mode shapes" or "normal modes"

A single mode behaves like a 1DOF system

5.4.3 Counting DOF and vibration modes

Approach #1   Count # coords needed to describe motion

Approach #2   Let: r = # rigid bodies
p = # particles
c = # constraints

Then:

For 3D system $\# DOF = 6r + 3p - c$
For 2D system $\# DOF = 3r + 2p - c$

vibration modes = # DOF − # "rigid body modes"

"Rigid body mode" is non vibration motion
– translation or rotation @ const speed

Examples of 3D constraints

Pinned joint

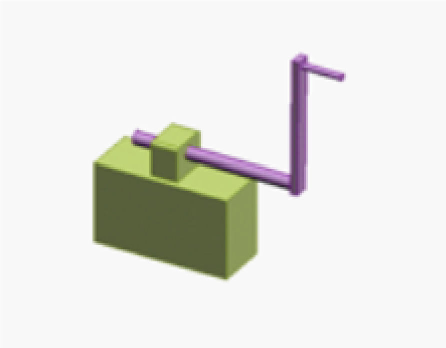
(5 constraints – prevents all motion, and prevents rotation about two axes)

Swivel joint

4 constraints (prevents all motion, prevents rotation about 1 axis)

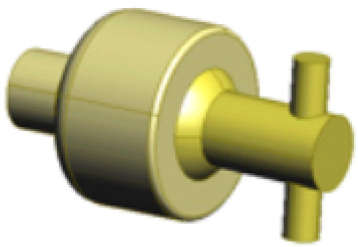
Roller bearing

(5 constraints – prevents all motion, and prevents rotation about two axes)



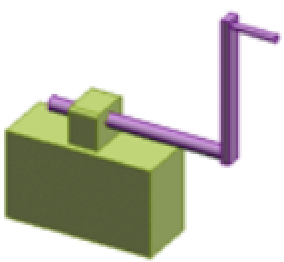
Ball and socket joint

3 constraints – prevents all motion.

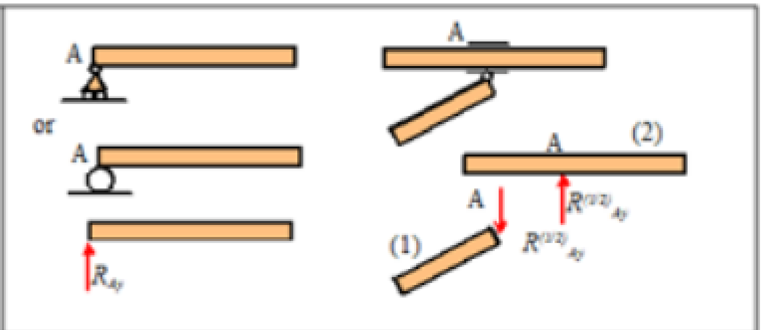
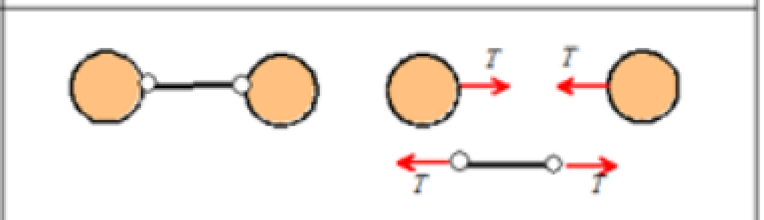
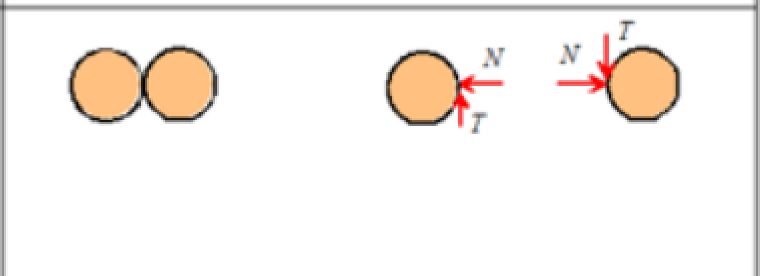
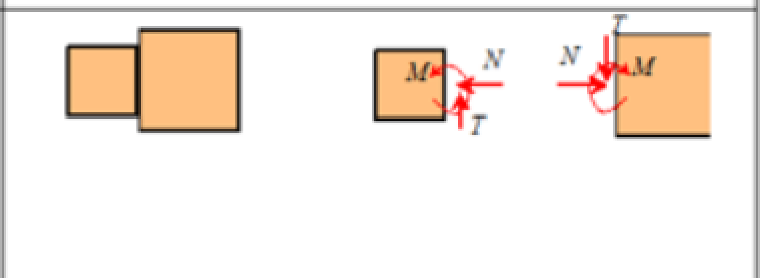
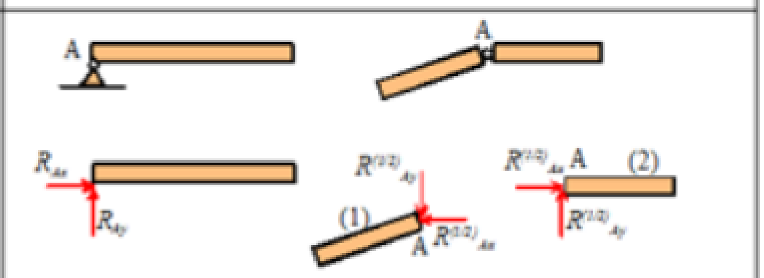


Sleeve

(4 constraints – prevents motion in two directions, and prevents rotation about two axes)



Examples of 2D constraints

Roller joint 1 constraint (prevents motion in one direction)	
Rigid (massless) link (if the link has mass, it should be represented as a rigid body) 1 constraint (prevents relative motion parallel to link)	
Nonconformal contact (two bodies meet at a point) **No friction or slipping:** 1 constraint (prevents interpenetration) **Sticking friction** 2 constraints (prevents relative motion	
Conformal contact (two rigid bodies meet along a line) **No friction or slipping:** 2 constraint (prevents interpenetration and rotation) **Sticking friction** 3 constraints (prevents relative motion)	
Pinned joint (generally only applied to a rigid body, as it would stop a particle moving completely) 2 constraints (prevents motion horizontally and vertically)	

5.4.4 Examples of counting DOF & vibration modes

5.4.4(a) Example: Find the number of DOF and vibration modes for the 'hopping robot' shown in the figure

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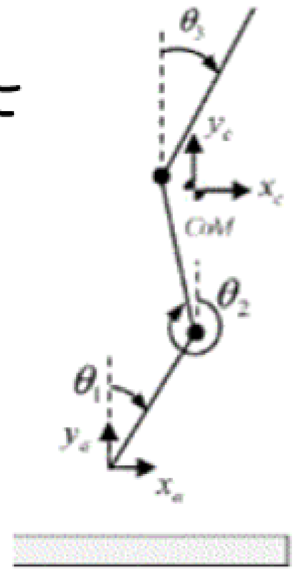
Approach #1 : $\{x_c, y_c, \theta_1, \theta_2, \theta_3\}$ are DOF

$\Rightarrow$ $\boxed{5 \text{ DOF}}$



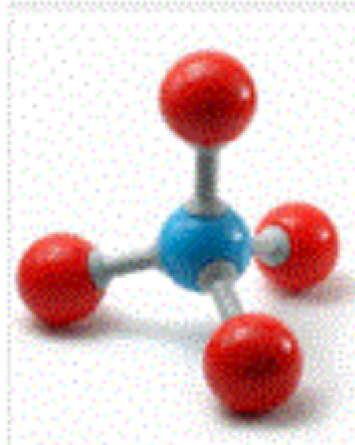
Approach #2 : $r = 3$ $p = 0$

Constraints: 2 pin joints, with 2 constraints each $\Rightarrow c = 4$

$$\boxed{\# \text{ DOF} = 3r + 2p - c = 9 - 4 = 5}$$

3 rigid body modes (2 translation, 1 rotation) $\Rightarrow$ $\boxed{\# \text{ vibe modes} = 2}$

5.4.4(b) Example: Find the number of DOF and vibration modes for the methane molecule shown in the figure



Approach 2

$r = 0$

$p = 5$

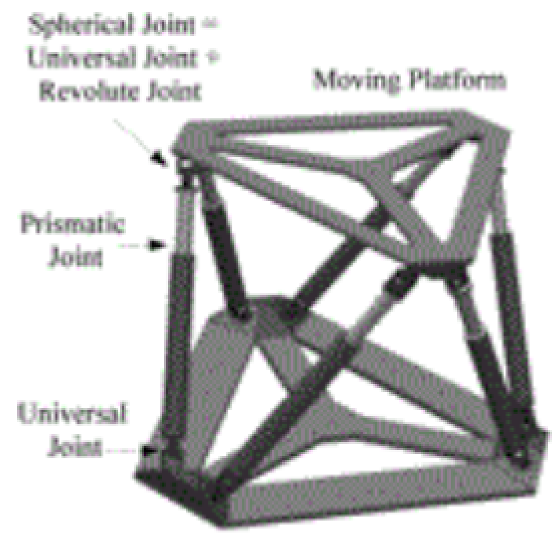
$c = 0$

$\Rightarrow \boxed{\# \text{ DOF} = 6r + 3p - c = 15}$

rigid body modes = 6
(3 translation, 3 rotation)

$\Rightarrow \boxed{\# \text{ vibration modes} = 9}$

5.4.4(c) Example: Find the number of DOF and vibration modes for the 'Stewart platform' shown in the figure



Base is fixed

\# rigid bodies :

Platform + 12 members = 13

Constraints :

6 spherical joints , 3 constraints each

6 universal joints , 4 constraints each

6 prismatic joints , 5 constraints each

$$\boxed{\# \text{ DOF} = 6r + 3p - c = 6(13 - 3 - 4 - 5) = 6}$$

No rigid body modes (base prevents any motion @ const speed)

$\Rightarrow$ $\boxed{\# \text{ vibration modes} = 6}$